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Stopping Times and an Extension of Stochastic Integrals in the Plane

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Let $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$ be a probability space with an increasing family of sub-σ-fields $\{\mathfrak{F}_z, z \in D\}$, where $D = [0, \infty) \times [0, \infty)$, satisfying the usual conditions. In this paper, the stochastic integral with respect to an $\mathfrak{F}_z$-adapted 2-parameter Brownian motion for integrand processes in the class $\mathfrak{L}_2(\mathfrak{F}_z)$ is extended, by means of truncations by $\{0, 1\}$-valued 2-parameter stopping times, to integrand processes that are $\mathfrak{F}_z$-adapted and continuous. The stochastic integral in the plane thus extended resembles a locally square integrable martingale in the 1-parameter setting. A definition of a parameter-space valued, i.e., D-valued, stopping time is also given and its characteristic process is related to a $\{0, 1\}$-valued 2-parameter stopping time.

1. INTRODUCTION

Let σ be a stopping time relative to an increasing family of sub-σ-fields $\{\mathfrak{F}_t, t \in [0, \infty)\}$ in a probability space $(\Omega, \mathfrak{F}, P)$; i.e., σ is a $[0, \infty]$-valued random variable such that $\{\omega \in \Omega; \sigma(\omega) \leqslant t\} \in \mathfrak{F}_t$ for every $t \in [0, \infty]$. Let us define a stochastic process T by

$$\begin{aligned} T(t, \omega) &= 1 \qquad \text{for} \quad t \in [0, \sigma(\omega)) \\ &= 0 \qquad \text{for} \quad t \in [\sigma(\omega), \infty) \end{aligned} \tag{1.1}$$

and call it the characteristic process of σ. This process plays the role of stochastic truncation in the theory of stochastic integrals. For instance, let Φ be a stochastic process which is progressively measurable relative to $\{\mathfrak{F}_t\}$ and has locally square integrable sample functions, i.e., for every $t \in [0, \infty)$

$$\int_0^t \Phi^2(s, \omega)\, ds < \infty \qquad \text{for} \quad \omega \in \Omega.$$

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With an increasing sequence of finite stopping times $\{\sigma_n, n=1,2,...\}$ with $\sigma_n(\omega)\uparrow\infty$ as $n\to\infty$ for $\omega\in\Omega$ given by

$$\sigma_n(\omega)=\inf\left\{t\in[0,\infty); \int_0^t \Phi^2(s,\omega)\,ds\geqslant n\right\}$$
$$\wedge\, n, \qquad \text{for} \quad n=1,2,..., \tag{1.2}$$

for each n, we define a stochastic process Φ_n by

$$\Phi_n(t,\omega)=T_n(t,\omega)\,\Phi(t,\omega),$$

where T_n is the characteristic process of σ_n. Then

$$\mathbb{E}\left[\int_0^t \Phi_n^2(s,\cdot)\,ds\right]<\infty \qquad \text{for every} \quad t\in[0,\infty)$$

so that the stochastic integral $I(\Phi_n)$ of Φ_n with respect to an $\mathfrak{F}_t$-adapted Brownian motion B, i.e.,

$$I(\Phi_n)(t)=\int_0^t \Phi_n(s)\,dB(s), \qquad \text{for} \quad t\in[0,\infty),$$

exists and is a square integrable martingale with continuous sample functions vanishing at $t=0$. The stochastic integral $I(\Phi)$ of Φ with respect to B is then defined to be the unique stochastic process X such that

$$X(t\wedge\sigma_n)=I(\Phi_n)(t) \qquad \text{for} \quad t\in[0,\infty) \text{ and } n=1,2,....$$

Thus defined, $I(\Phi)$ is a locally square integrable martingale with continuous sample functions vanishing at $t=0$.

Let us consider 2-parameter stochastic processes on the domain $D=[0,\infty)\times[0,\infty)$. We introduce a partial order $\prec$ in D by defining for $z=(s,t)$ and $z'=(s',t')$ in D:

$$z\prec z' \qquad \text{when} \qquad s\leqslant s' \quad \text{and} \quad t\leqslant t'.$$

We write

$$z\lll z' \qquad \text{when} \qquad s<s' \quad \text{and} \quad t<t'.$$

For $z=(s,t)$ in D we write $[0,z]$ for $[0,s]\times[0,t]\subset D$.

We write $(\Omega,\mathfrak{F},P;\mathfrak{F}_z)$ when

1. $(\Omega,\mathfrak{F},P)$ is a complete probability space,

2. $\{\mathfrak{F}_z, z\in D\}$ is an increasing family of sub-σ-fields of $\mathfrak{F}$ in the sense that $\mathfrak{F}_z\subset\mathfrak{F}_{z'}$ for $z\prec z'$,

3. $\mathfrak{F}_0$ contains all the null sets of $(\Omega, \mathfrak{F}, P)$,
4. $\{\mathfrak{F}_z, z \in D\}$ is a right continuous family in the sense that

$$\mathfrak{F}_z = \bigcap_{z \ll z'} \mathfrak{F}_{z'} \qquad \text{for} \quad z \in D,$$

5. for every $z = (s, t) \in D$, $\mathfrak{F}_{s,\cdot} \equiv \sigma(\bigcup_{v \in [0,\infty)} \mathfrak{F}_{s,v})$ and $\mathfrak{F}_{\cdot,t} \equiv \sigma(\bigcup_{u \in [0,\infty)} \mathfrak{F}_{u,t})$ are conditionally independent relative to $\mathfrak{F}_z$.

A 2-parameter stochastic process X on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$ and D is said to be $\mathfrak{F}_z$-adapted if for every $z \in D$, $X(z, \cdot)$ is $\mathfrak{F}_z$-measurable. X is called a measurable process if as a real valued function on $D \times \Omega$ it is $\sigma(\mathfrak{B}(D) \times \mathfrak{F})$-measurable, where $\mathfrak{B}(D)$ is the σ-field of Borel sets in D. X is said to be progressively measurable relative to $\{\mathfrak{F}_z\}$ if for every $z \in D$ the restriction of X to $[0, z] \times \Omega$ is $\sigma(\mathfrak{B}([0, z]) \times \mathfrak{F}_z)$-measurable, where $\mathfrak{B}([0, z])$ is the σ-field of Borel sets in $[0, z]$.

DEFINITION 1 (Wong and Zakai [5]). Given $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$. A function T on $D \times \Omega$ is called a 2-parameter stopping time relative to $\{\mathfrak{F}_z\}$ if

1. T is an $\mathfrak{F}_z$-adapted and measurable 2-parameter stochastic process,
2. the range of T is contained in the set $\{0, 1\}$,
3. T is nonincreasing relative to $z \in D$ in the sense that $z \prec z'$ implies $T(z, \omega) \geqslant T(z', \omega)$ for every $\omega \in \Omega$.

We remark that the 2-parameter stopping time as defined above is the 2-parameter analog of the characteristic process of a 1-parameter stopping time rather than the analog of a 1-parameter stopping time itself. We remark also that conditions 1 and 2 in Definition 1 imply that a 2-parameter stopping time always has a progressively measurable version. This follows from the following proposition, which we state without giving the proof.

PROPOSITION 1. *Let Φ be an $\mathfrak{F}_z$-adapted and measurable 2-parameter stochastic process on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$ and D which is integrable on $[0, z] \times \Omega$ for every $z \in D$. Then Φ has a progressively measurable version relative to $\{\mathfrak{F}_z\}$. In fact there exists a null set N in $(\Omega, \mathfrak{F}, P)$ such that if we define*

$$\begin{aligned} \Psi(s, t, \omega) &= \limsup_{h \downarrow 0} h^{-2} \int_{[s-h,s] \times [t-h,t]} \Phi(u, v, \omega) \\ &\qquad \times m_L(d(u, v)) \qquad \text{for} \quad (s, t) \in D \text{ and } \omega \in N^c, \\ &= 0 \qquad \text{for} \quad (s, t) \in D \text{ and } \omega \in N, \end{aligned}$$

where m_L is the Lebesgue measure, then Ψ is progressively measurable relative to $\{\mathfrak{F}_z\}$ and $\Psi(\cdot, \cdot, \omega) = \Phi(\cdot, \cdot, \omega)$ for $\omega \in N^c$.

DEFINITION 2. By a 2-parameter martingale relative to $\{\mathfrak{F}_z\}$ we mean a 2-parameter stochastic process M on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$ and D satisfying the following conditions:

1. M is $\mathfrak{F}_z$-adapted,
2. $\mathbb{E}[|M_z|] < \infty$ for $z \in D$,
3. $\mathbb{E}[M_{z'} \mid \mathfrak{F}_z] = M_z$ for $z \prec z'$,
4. a.e. sample function of M is right continuous on D in the sense that

$$\lim_{z' \to z, z \prec z'} M(z', \omega) = M(z, \omega) \qquad \text{for all } z \in D \text{ for a.e. } \omega \in \Omega.$$

We write $\mathfrak{M}(\mathfrak{F}_z)$ for the class of 2-parameter martingales relative to $\{\mathfrak{F}_z\}$ that vanish on the boundary of D. We write $\mathfrak{M}_2(\mathfrak{F}_z)$ for the subclass of $\mathfrak{M}(\mathfrak{F}_z)$ consisting of square integrable 2-parameter martingales, i.e., those M in $\mathfrak{M}(\mathfrak{F}_z)$ which satisfy,

5. $\mathbb{E}[M_z^2] < \infty$ for $z \in D$.

Finally, $\mathfrak{M}_2^c(\mathfrak{F}_z)$ designates the subclass of $\mathfrak{M}_2(\mathfrak{F}_z)$ consisting of those members of $\mathfrak{M}_2(\mathfrak{F}_z)$ whose sample functions are continuous on D for a.e. $\omega \in \Omega$.

Let $\mathfrak{L}_2(\mathfrak{F}_z)$ be the class of $\mathfrak{F}_z$-adapted and measurable 2-parameter stochastic processes Φ on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$ and D satisfying

$$\mathbb{E}\left[\int_{[0,z]} \Phi^2(z, \cdot)\, m_{\mathrm{L}}(dz)\right] < \infty \qquad \text{for every } z \in D. \tag{1.3}$$

By Proposition 1, such Φ has a progressively measurable version relative to $\{\mathfrak{F}_z\}$ and thus its stochastic integral with respect to an $\mathfrak{F}_z$-adapted 2-parameter Brownian motion B on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$ and D,

$$M(z, \omega) = \int_{[0,z]} \Phi(\varphi, \omega)\, dB(\varphi, \omega) \qquad \text{for} \quad z \in D,$$

can be defined in a manner analogous to that for the stochastic integral in the 1-parameter setting and in fact M is a member of $\mathfrak{M}_2^c(\mathfrak{F}_z)$ (see, for instance, Cairoli and Walsh [1] and Wong and Zakai [4]).

Consider now the class $\mathfrak{L}_2^{\mathrm{loc}}(\mathfrak{F}_z)$ of $\mathfrak{F}_z$-adapted and measurable 2-parameter stochastic processes satisfying

$$\int_{[0,z]} \Phi^2(z, \omega)\, m_{\mathrm{L}}(dz) < \infty \qquad \text{for} \quad \text{a.e. } \omega \in \Omega \text{ for every } z \in D. \tag{1.4}$$

The attempt by Wong and Zakai to extend the definition of the stochastic integral from the class $\mathfrak{L}_2(\mathfrak{F}_z)$ to the class $\mathfrak{L}_2^{\mathrm{loc}}(\mathfrak{F}_z)$ was abandoned since the 2-parameter analogs of the stopping times (1.2) (or rather their characteristic processes) were not appropriate for the purpose (see, [6, p. 771]).

In Section 2 of the present paper we show that by means of truncations by 2-parameter stopping times, the stochastic integral can be extended from the class $\mathfrak{L}_2(\mathfrak{F}_z)$ to the class of continuous stochastic processes, and that the stochastic integral thus extended is a 2-parameter analog ofa locally square integrable martingale in the 1-parameter setting.

A stopping time in the 1-parameter setting is a $[0, \infty]$-valued, i.e., a parameter-space valued, random variable. In Section 3 we define a D-valued stopping time and relate its characteristic process to a 2-parameter stopping time as given in Definition 1.

2. The Stochastic Integral of Continuous 2-Parameter Stochastic Processes

Throughout Section 2 we assume that $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$ satisfying the conditions in Section 1 is fixed.

Definition 3. Let $\{T_n, n = 1, 2,...\}$ be a sequence of 2-parameter stopping times relative to $\{\mathfrak{F}_z\}$. For each $z \in D$, let

$$\Omega_n(z) = \{\omega \in \Omega; T_n(z, \omega) = 1\} \qquad \text{for} \quad n = 1, 2,.... \tag{2.1}$$

We say that the sequence $\{T_n\}$ is increasing if with a null set Λ_0 in $(\Omega, \mathfrak{F}, P)$ and $\Omega_0 = \Lambda_0^c$ we have

$$\Omega_n(z) \uparrow \Omega_0 \qquad \text{as} \quad n \to \infty \text{ for every } z \in D. \tag{2.2}$$

According to Proposition 1, by redefining T_n on $D \times \Lambda_n$, where Λ_n is a null set in $(\Omega, \mathfrak{F}, P)$, to be identically equal to 0, T_n can be made a progressively measurable 2-parameter stopping time relative to $\{\mathfrak{F}_z\}$. Then with Λ_0 as in Definition 3 but with $\Omega_0 = (\bigcup_{n=0}^{\infty} \Lambda_n)^c$, the condition (2.2) is satisfied by the sequence of redefined, and progressively measurable, 2-parameter stopping times.

Definition 4. A 2-parameter stochastic process X is called a locally square integrable martingale relative to $\{\mathfrak{F}_z\}$ if there exist an increasing sequence $\{T_n, n = 1, 2,...\}$ of progressively measurable 2-parameter stopping times relative to $\{\mathfrak{F}_z\}$ and a sequence $\{M_n, n = 1, 2,...\}$ in $\mathfrak{M}_2(\mathfrak{F}_z)$ such that for every $z_0 \in D$ we have

$$X(z, \omega) = M_n(z, \omega) \qquad \text{for} \quad (z, \omega) \in [0, z_0] \times (\Omega_n(z_0) - \Lambda), \tag{2.3}$$

where $\Omega_n(z_0)$ is the z_0-section of the support of T_n, i.e.,

$$\Omega_n(z_0) = \{\omega \in \Omega;\ T_n(z_0, \omega) = 1\} \tag{2.4}$$

and Λ is a null set in $(\Omega, \mathfrak{F}, P)$.

PROPOSITION 2. *Let Φ be an $\mathfrak{F}_z$-adapted 2-parameter stochastic process every sample function of which is continuous. Define a 2-parameter stochastic process φ by*

$$\varphi(z, \omega) = \sup_{\zeta < z} |\Phi(\zeta, \omega)| \qquad \textit{for} \quad (z, \omega) \in D \times \Omega.$$

For each positive integer n define a function T_n on $D \times \Omega$ by

$$\begin{aligned} T_n(z, \omega) &= 1 \qquad \textit{when} \quad \varphi(z, \omega) < n \\ &= 0 \qquad \textit{when} \quad \varphi(z, \omega) \geqslant n. \end{aligned}$$

Then $\{T_n, n = 1, 2, \ldots\}$ is an increasing sequence of progressively measurable 2-parameter stopping times relative to $\{\mathfrak{F}_z\}$.

Proof. The $\mathfrak{F}_z$-adaptedness and the continuity of every sample function of Φ imply that Φ is progressively measurable relative to $\{\mathfrak{F}_z\}$. (Actually, in the Lemma, Section 3, we shall show that under a weaker continuity condition, an $\mathfrak{F}_z$-adapted 2-parameter stochastic process is progressively measurable relative to $\{\mathfrak{F}_z\}$.) The continuity of the sample functions of Φ implies that φ is an $\mathfrak{F}_z$-adapted 2-parameter stochastic process every sample function of which is continuous. Thus φ too is progressively measurable relative to $\{\mathfrak{F}_z\}$.

Let us show that for every n, T_n is a progressively measurable 2-parameter stopping time relative to $\{\mathfrak{F}_z\}$. Now the range of T_n is contained in the set $\{0, 1\}$. The progressive measurability of T_n then follows from that of φ since for every $z \in D$

$$\begin{aligned} &\{(\zeta, \omega) \in [0, z] \times \Omega;\ T_n(\zeta, \omega) = 1\} \\ &\quad = \{(\zeta, \omega) \in [0, z] \times \Omega;\ \varphi(z, \omega) < n\} \in \sigma(\mathfrak{B}([0, z]) \times \mathfrak{F}_z). \end{aligned}$$

To complete the proof that T_n is a 2-parameter stopping time, it remains to show that T_n is nonincreasing relative to $z \in D$. For this, note first that $\varphi(z, \omega) \leqslant \varphi(z', \omega)$ when $z < z'$ for every $\omega \in \Omega$. Now if $T_n(z, \omega) = 1$, then $T_n(z, \omega) \geqslant T_n(z', \omega)$ for any z'. On the other hand if $T_n(z, \omega) = 0$ then $\varphi(z, \omega) \geqslant n$ and thus $\varphi(z', \omega) \geqslant n$ also for $z < z'$. This implies that $T_n(z', \omega) = 0$ and therefore $T_n(z, \omega) = T_n(z', \omega)$ for $z < z'$. Therefore $T_n(z, \omega) \geqslant T_n(z', \omega)$ for $z < z'$.

Finally, to show that $\{T_n, n = 1, 2,...\}$ is an increasing sequence, let $z \in D$ and consider

$$\Omega_n(z) = \{\omega \in \Omega;\, T_n(z, \omega) = 1\} = \{\omega \in \Omega;\, \varphi(z, \omega) < n\}.$$

Clearly $\Omega_n(z)\uparrow$ as $n \to \infty$. To show that $\Omega_n(z) \uparrow \Omega$ as $n \to \infty$, let $\omega_0 \in \Omega$. From the continuity of $\Phi(\cdot, \omega_0)$ on D

$$\varphi(z, \omega_0) = \sup_{\zeta < z} |\Phi(\zeta, \omega_0)| < n_0$$

for sufficiently large n_0. Then $T_{n_0}(z, \omega_0) = 1$ so that $\omega_0 \in \Omega_{n_0}(z)$. Thus $\bigcup_n \Omega_n(z) = \Omega$, i.e., $\Omega_n(z) \uparrow \Omega$ as $n \to \infty$.

2.1. *The Stochastic Integral of a Continuous Process*

Let Φ be an $\mathfrak{F}_z$-adapted 2-parameter stochastic process every sample function of which is continuous. Let $\{T_n, n = 1, 2,...\}$ be the increasing sequene of progressively measurable 2-parameter stopping times relative to $\{\mathfrak{F}_z\}$ as defined in Proposition 2. Then for each n, T_n is a progressively measurable 2-parameter stochastic process relative to $\{\mathfrak{F}_z\}$ and is bounded by n on $D \times \Omega$ so that its stochastic integral with respect to the $\mathfrak{F}_z$-adapted 2-parameter Brownian motion B,

$$M_n(z, \omega) = \int_{[0,z]} \Phi(\zeta, \omega)\, T_n(\zeta, \omega)\, dB(\zeta, \omega) \qquad \text{for} \quad z \in D, \tag{2.5}$$

is defined and is a member of $\mathfrak{M}_2^c(\mathfrak{F}_z)$.

Let $z_m = (m, m) \in D$ for $m = 1, 2,...$ and define

$$M_{n,m}(z, \omega) = \int_{[0,z_m]} \chi_{[0,z]}(\zeta)\, \Phi(\zeta, \omega)\, T_n(\zeta, \omega)\, dB(\zeta, \omega)$$
$$\text{for} \quad z \in [0, z_m],$$

where $\chi_{[0,z]}$ is the characteristic function of the set $[0, z]$. Thus $M_{n,m}$ is the restriction of M_n to $[0, z_m]$ so that

$$M_{n,m}(z, \omega) = M_n(z, \omega) \qquad \text{for} \quad (z, \omega) \in [0, z_m] \times \Omega.$$

With m fixed, let

$$\Omega_n(z_m) = \{\omega \in \Omega;\, T_n(z_m, \omega) = 1\} \qquad \text{for} \quad n = 1, 2,\ldots.$$

Since $\Omega_{n_1}(z_m) \subset \Omega_{n_2}(z_m)$ and the integrands of $M_{n_1,m}$ and $M_{n_2,m}$ are identical on $[0, z_m] \times \Omega_{n_1}(z_m)$ for $n_1 < n_2$, we have

$$M_{n_1,m}(z, \omega) = M_{n_2,m}(z, \omega) \qquad \text{for} \quad (z, \omega) \in [0, z_m] \times (\Omega_{n_1}(z_m) - \Lambda_{n_1,n_2,m}),$$

where $\Lambda_{n_1,n_2,m}$ is a null set in $(\Omega, \mathfrak{F}, P)$. Let $\Lambda_m = \bigcup_{n_1,n_2} \Lambda_{n_1,n_2,m}$. Since $\Omega_n(z_m) \uparrow \Omega$ as $n \to \infty$, there exists a real valued function X_m defined on $[0, z_m] \times \Omega$ such that for every n we have

$$\begin{aligned} X_m(z, \omega) &= M_{n,m}(z, \omega) \\ &= M_n(z, \omega) \qquad \text{for} \quad (z, \omega) \in [0, z_m] \times (\Omega_n(z_m) - \Lambda_m). \end{aligned} \tag{2.6}$$

Consider now X_m defined on $[0, z_m] \times \Omega$ for $m = 1, 2, \ldots$. Let $\Lambda = \bigcup_m \Lambda_m$. We proceed to show that for $m_1 < m_2$ we have

$$X_{m_1}(z, \omega) = X_{m_2}(z, \omega) \qquad \text{for} \quad (z, \omega) \in [0, z_{m_1}] \times (\Omega - \Lambda). \tag{2.7}$$

Now, by (2.6) for m_1 and m_2 we have, for every n,

$$\begin{aligned} X_{m_1}(z, \omega) = X_{m_2}(z, \omega) \qquad \text{for} \quad & (z, \omega) \in [0, z_{m_1}] \\ & \times (\Omega_n(z_{m_1}) \cap \Omega_n(z_{m_2}) - \Lambda). \end{aligned}$$

Since this equality holds for every n and since $\Omega_n(z_{m_i}) \uparrow \Omega$ for $i = 1, 2$ when $n \to \infty$, we have (2.7).

Therefore there exists a real valued function X defined on $D \times \Omega$ such that for every m

$$X(z, \omega) = X_m(z, \omega) \qquad \text{for} \quad (z, \omega) \in [0, z_m] \times (\Omega - \Lambda),$$

or, recalling (2.6), for every m and n

$$X(z, \omega) = M_n(z, \omega) \qquad \text{for} \quad (z, \omega) \in [0, z_m] \times (\Omega_n(z_m) - \Lambda). \tag{2.8}$$

This real valued function X on $D \times \Omega$, uniquely defined except on a null set Λ of $(\Omega, \mathfrak{F}, P)$, is a locally square integrable martingale relative to $\{\mathfrak{F}_z\}$. We define the stochastic integral of Φ with respect to B to be X, i.e.,

$$\int_{[0,z]} \Phi(\zeta, \omega)\, dB(\zeta, \omega) = X(z, \omega) \qquad \text{for} \quad z \in D.$$

3. 2-Parameter Stopping Times with Values in the Parameter Space

As before, the domain of definition of a 2-parameter stochastic process is $D = [0, \infty) \times [0, \infty)$. For $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$, however, we assume only that $\{\mathfrak{F}_z, z \in D\}$ is an increasing family of sub-σ-fields of $\mathfrak{F}$ relative to the partial order $\prec$ in D. Let θ be an increasing path in D in the sense that θ is a continuous transformation of $[0, \infty)$ into D such that $\theta(r) \prec \theta(r')$ for $r < r'$. Wong and Zakai [4] have shown that a 2-parameter stochastic process M on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$ and D is a 2-parameter martingale relative to $\{\mathfrak{F}_z\}$ if and only if $\{M_{\theta(r)}, r \in [0, \infty)\}$ is a martingale relative to $\{\mathfrak{F}_{\theta(r)}, r \in [0, \infty)\}$ for every increasing path θ. Suppose that for each increasing path θ a D-valued stopping time σ_θ relative to $\{\mathfrak{F}_{\theta(r)}, r \in [0, \infty)\}$ is given. We shall find a compatibility condition among the σ_θ's that will ensure that a D-valued 2-parameter stopping timerelative to $\{\mathfrak{F}_z\}$ is defined by the σ_θ's. To facilitate the discussion we sharpen the definition of an increasing path as follows:

DEFINITION 5. Let Θ be the collection of all transformations θ of $[0, \infty)$ into D satisfying

1. θ is a continuous transformation of $[0, \infty)$ into D,
2. θ is strictly increasing in the sense that

$$r < r' \Rightarrow \theta(r) \prec \theta(r') \qquad \text{and} \qquad \theta(r) \neq \theta(r'),$$

3. $\theta(0) = (0, 0)$,
4. $\mathfrak{R}(\theta)$, the range of θ, is an unbounded set in D.

For $\theta \in \Theta$ we let $\theta(\infty) = \lim_{r \to \infty} \theta(r)$, which always exists as an improper point of the types (a, ∞), (∞, b), and (∞, ∞) with $a, b \in [0, \infty)$ because of conditions 1, 2, and 4 of Definition 5.

DEFINITION 6. Given $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$, a collection $\mathfrak{S} = \{\sigma_\theta, \theta \in \Theta\}$ of transformations of Ω into D is called a D-valued 2-parameter stopping time relative to $\{\mathfrak{F}_z\}$ if

1. $\sigma_\theta(\Omega) \subset \mathfrak{R}(\theta)$ for every $\theta \in \Theta$,
2. for every $\theta \in \Theta$ and $z \in \mathfrak{R}(\theta)$

$$\{\omega \in \Omega; \sigma_\theta(\omega) \prec z\} \in \mathfrak{F}_z, \tag{3.1}$$

3. for every $z \in D$ and $\omega \in \Omega$, whenever $\theta_1, \theta_2 \in \Theta$ and $z \in \mathfrak{R}(\theta_1) \cap \mathfrak{R}(\theta_2)$, then

$$\sigma_{\theta_1}(\omega) \prec z \Rightarrow \sigma_{\theta_2}(\omega) \prec z, \tag{3.2}$$

or equivalently

$$z \prec \sigma_{\theta_2}(\omega), \qquad z \neq \sigma_{\theta_2}(\omega) \Rightarrow z \prec \sigma_{\theta_1}(\omega), \qquad z \neq \sigma_{\theta_1}(\omega). \tag{3.2$'$}$$

By the characteristic process of $\mathfrak{S}$ we mean the function $I_{\mathfrak{S}}$ on $D \times \Omega$ defined by

$$\begin{aligned} I_{\mathfrak{S}}(z, \omega) &= 0 \qquad \text{when} \quad \sigma_\theta(\omega) \prec z \\ &= 1 \qquad \text{when} \quad z \prec \sigma_\theta(\omega) \text{ and } z \neq \sigma_\theta(\omega) \end{aligned} \tag{3.3}$$

by means of an arbitrary $\theta \in \Theta$ with $z \in \mathfrak{R}(\theta)$.

We remark that the definition $I_{\mathfrak{S}}$ above does not depend on the particular $\theta \in \Theta$ used in defining its value at (z, ω). This is a consequence of condition 3. Condition 2 is equivalent to stating that for every $\theta \in \Theta$, σ_θ is an $\mathfrak{R}(\theta)$-valued stopping time relative to $\{\mathfrak{F}_{\theta(r)}, r \in [0, \infty)\}$. We shall show that $I_{\mathfrak{S}}$ is progressively measurable relative to $\{\mathfrak{F}_z\}$ and is also a 2-parameter stopping time relative to $\{\mathfrak{F}_z\}$ in the sense of Definition 1. For this we need the following lemma.

LEMMA. *Let X be an $\mathfrak{F}_z$-adapted 2-parameter stochastic process on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$ and D. Suppose that for every $\omega \in \Omega$ the sample function $X(\cdot, \omega)$ has the property that at each $z \in D$, for every $\theta \in \Theta$ with $z \in \mathfrak{R}(\theta)$, we have*

$$\lim_{z' \to z, z' \in \mathfrak{R}(\theta), z < z'} X(z', \omega) = X(z, \omega). \tag{3.4}$$

Then X is progressively measurable relative to $\{\mathfrak{F}_z\}$.

Proof. Let $z = (s, t) \in D$ be fixed. For every positive integer n, decompose the rectangle $[0, s) \times [0, t)$ into $2^n \times 2^n$ equal subrectangles of the same type and define a real valued function $X^{(n)}$ on $[0, s] \times [0, t] \times \Omega$ by

$$\begin{aligned}
X^{(n)}(u, v, \omega) &= X((j+1)\, 2^{-n}s, (k+1)\, 2^{-n}t, \omega) \\
&\qquad \text{for} \quad (u, v) \in [j2^{-n}s, (j+1)\, 2^{-n}s) \times [k2^{-n}t, (k+1)\, 2^{-n}t) \\
&= X((j+1)\, 2^{-n}s, t, \omega) \\
&\qquad \text{for} \quad u \in [j2^{-n}s, (j+1)\, 2^{-n}s) \text{ and } v = t \\
&= X(s, (k+1)\, 2^{-n}t, \omega) \\
&\qquad \text{for} \quad u = s \text{ and } v \in [k2^{-n}t, (k+1)\, 2^{-n}t) \\
&= x(s, t, \omega) \qquad \text{for} \quad (u, v) = (s, t),
\end{aligned}$$

where j, $k = 0, 1, 2, \ldots, 2^n - 1$. Then $X^{(n)}$ is measurable relative to $\sigma(\mathfrak{B}([0, z]) \times \mathfrak{F}_z)$.

Let $(u, v) \in [0, s) \times [0, t)$. Then for each n there exists a unique subrectangle $[s'_n, s''_n) \times [t'_n, t''_n)$ in the nth decomposition of $[0, s) \times [0, t)$ that contains (u, v). Consider a member θ of Θ that connects all (s''_n, t''_n), $n = 1, 2, \ldots$, to (u, v). Then by (3.4) we have

$$\lim_{n \to \infty} X^{(n)}(u, v, \omega) = \lim_{n \to \infty} X(s''_n, t''_n, \omega) = X(u, v, \omega).$$

The same holds for (s, v) with $v \in [0, t]$ and (u, t) with $u \in [0, s]$. Thus X restricted to $[0, s] \times [0, t]$ is measurable relative to $\sigma(\mathfrak{B}([0, z]) \times \mathfrak{F}_z)$. This proves the progressive measurability of X relative to $\{\mathfrak{F}_z\}$.

THEOREM 1. *Let $I_{\mathfrak{S}}$ be the characteristic process of a D-valued 2-parameter stopping time $\mathfrak{S} = \{\sigma_\theta, \theta \in \Theta\}$ on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$. Then*

1. *$I_{\mathfrak{S}}$ is a progressively measurable 2-parameter stochastic process relative to $\{\mathfrak{F}_z\}$,*

2. *$I_{\mathfrak{S}}$ is a 2-parameter stopping time relative to $\{\mathfrak{F}_z\}$ in the sense of Definition* 1,

3. *for every $\theta \in \Theta$ and $\omega \in \Omega$, $I_{\mathfrak{S}}(\cdot, \omega)$ as a function of $z \in \mathfrak{R}(\theta)$ has the property that there exists a unique $\zeta_{\theta,\omega} \in \mathfrak{R}(\theta)$ such that*

$$z \in \mathfrak{R}(\theta), \qquad \zeta_{\theta,\omega} \prec z \Rightarrow I_{\mathfrak{S}}(z, \omega) = 0,$$
$$z \in \mathfrak{R}(\theta), \qquad z \prec \zeta_{\theta,\omega}, \qquad z \neq \zeta_{\theta,\omega} \Rightarrow I_{\mathfrak{S}}(z, \omega) = 1.$$

Proof. To show that $I_{\mathfrak{S}}$ is $\mathfrak{F}_z$-adapted, let $z \in D$ be fixed. Then with an arbitrary $\theta \in \Theta$ with $z \in \mathfrak{R}(\theta)$ we have, by (3.3) and (3.1) in Definition 6

$$\{\omega \in \Omega; I_{\mathfrak{S}}(z, \omega) = 0\} = \{\omega \in \Omega; \sigma_\theta(\omega) \prec z\} \in \mathfrak{F}_z.$$

Since the range of $I_{\mathfrak{S}}$ is contained in $\{0, 1\}$ this shows that $I_{\mathfrak{S}}$ is $\mathfrak{F}_z$-adapted. Definition (3.3) also shows that $I_{\mathfrak{S}}$ satisfies condition (3.4) in Lemma. Therefore $I_{\mathfrak{S}}$ is progressively measurable relative to $\{\mathfrak{F}_z\}$. Now that $I_{\mathfrak{S}}$ is progressively relative to $\{\mathfrak{F}_z\}$ it is in particular measurable. Therefore to show that $I_{\mathfrak{S}}$ is a 2-parameter stopping time relative to $\{\mathfrak{F}_z\}$ in the sense of Definition 1 it remains to verify that $I_{\mathfrak{S}}(z, \omega)$ is nonincreasing relative to $z \in D$ for each $\omega \in \Omega$. But this too is obvious from (3.3). Finally, part 3 is immediate from (3.3).

THEOREM 2. *Let I be a $\{0, 1\}$-valued and $\mathfrak{F}_z$-adapted 2-parameter stochastic process on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_z)$ and D. Assume that I has property* 3 *of Theorem* 1, *and for each $\theta \in \Theta$ define a transformation σ_θ of Ω into $\mathfrak{R}(\theta)$ by setting*

$$\sigma_\theta(\omega) = \zeta_{\theta,\omega}. \tag{3.5}$$

Then $\mathfrak{S} = \{\sigma_\theta, \theta \in \Theta\}$ is a D-valued 2-parameter stopping time relative to $\{\mathfrak{F}_z\}$ and the characteristic process $I_{\mathfrak{S}}$ of $\mathfrak{S}$ is identical with I.

Proof. Condition 1 of Definition 6 is obviously satisfied by σ_θ for every $\theta \in \Theta$. To verify condition 2 of Definition 6, note that for $\theta \in \Theta$ and $z \in \mathfrak{R}(\theta)$ we have

$$\{\omega \in \Omega; \sigma_\theta(\omega) \prec z\} = \{\omega \in \Omega; \zeta_{\theta,\omega} \prec z\} = \{\omega \in \Omega; I(z, \omega) = 0\} \in \mathfrak{F}_z$$

by (3.5), property 3 of Theorem 1, and finally the fact that I is $\mathfrak{F}_z$ adapted.

To verify condition 3 of Definition 6, let $z \in D$, $\omega \in \Omega$, and $\theta_1, \theta_2 \in \Theta$ with $z \in \mathfrak{R}(\theta_1) \cap \mathfrak{R}(\theta_2)$. Suppose $\sigma_{\theta_1}(\omega) \prec z$, i.e., $\zeta_{\theta_1,\omega} \prec z$. Then $I(z, \omega) = 0$. Since $z \in \mathfrak{R}(\theta_2)$, $I(z, \omega) = 0$ implies $\zeta_{\theta_2,\omega} \prec z$, i.e., $\sigma_{\theta_2}(\omega) \prec z$. This shows that condition 3 of Definition 6 is satisfied. Therefore $\mathfrak{S}$ is a D-valued 2-parameter stopping time relative to $\{\mathfrak{F}_z\}$. The fact that $I_{\mathfrak{S}} = I$ follows immediately from (3.3) and (3.5).

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